Complex Numbers

1. If $z_1 = 4+3i$, $z_2 = 3-2i$, Find (a) $\frac{1}{z_1}+\frac{1}{z_2}$, (b) $\frac{1}{z_1 \cdot z_2}$

(a) $$\frac{1}{z_1}+\frac{1}{z_2} = \frac{1}{4+3i}+\frac{1}{3-2i} = \frac{3-2i+4+3i}{12-8i+9i-6i^2} = \frac{7+i}{18+i}$$

$$= \frac{7+i}{18+i} \times \frac{18-i}{18-i}$$

$$= \frac{126-7i+18i-i^2}{(18)^2-i^2}$$

$$= \frac{126+11i+1}{324+1}$$

$$= \frac{127+11i}{325}$$

$$= \frac{127}{325}+\frac{11}{325}i$$

(b) $$\frac{1}{z_1 \cdot z_2} = \frac{1}{(4+3i)(3-2i)}$$

$$= \frac{1}{12-8i+9i-6i^2}$$

$$= \frac{1}{18+i}$$

$$= \frac{1}{18+i} \times \frac{18-i}{18-i}$$

$$= \frac{18-i}{(18)^2-i^2}$$

$$= \frac{18-i}{324+1} = \frac{18-i}{325}$$

$$\frac{1}{z_1 \cdot z_2} = \frac{18}{325}-\frac{1}{325}i$$

② Express the foll^n in the form $x+iy$.

1) $\frac{4+2i}{(2+i)(2-i)}$.

Sol^n: $\frac{4+2i}{(2+i)(2-i)} = \frac{4+2i}{(2)^2-(i^2)} = \frac{4+2i}{4+1} = \frac{4+2i}{5} = \boxed{\frac{4}{5}+\frac{2}{5}i}$

③ If $\omega_1 = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$ and $\omega_2 = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$ show that $\omega_1^2 = \omega_2$

$\omega_1^2 = \left[-\frac{1}{2} - \frac{i\sqrt{3}}{2}\right]^2$

$= \left(-\frac{1}{2}\right)^2 - 2\left(-\frac{1}{2}\right)\cdot\frac{\sqrt{3}}{2}i + \left(\frac{\sqrt{3}}{2}i\right)^2$

$= \frac{1}{4} + \frac{\sqrt{3}}{2}i + \frac{3}{4}i^2$

$= \frac{1}{4} + \frac{\sqrt{3}}{2}i - \frac{3}{4}$ —— $\because (i^2 = -1)$

$= \frac{1-3}{4} + \frac{\sqrt{3}}{2}i$

$= \frac{2}{4} + \frac{\sqrt{3}}{2}i$

$= \frac{1}{2} + \frac{\sqrt{3}}{2}i$

$\therefore \omega_1^2 = \omega_2$.

④ Simplify : $1+i^{10}+i^{20}+i^{30}$

$1+i^{10}+i^{20}+i^{30} = 1+(i^2)^5+(i^2)^{10}+(i^2)^{15}$

$= 1+(-1)^5+(-1)^{10}+(-1)^{15}$

$= 1-1+1-1$

$= 0$.

5. If $(a-2bi)+(b-3ai)=5+2i$, find a & b.

Sol^n: $a-2bi+b-3ai=5+2i$

$(a+b)+i(-3a-2b)=5+2i$

Equating real & imaginary parts.

$a+b=5$ —— (1) and $-3a-2b=2$ —— (2)

Mul. eq^n (1) by 3 & solve eq^n (1) & (2)

$\therefore$ $\cancel{3}a+3b=15$

$-\cancel{3}a-2b=2$

$\boxed{b=17}$

Put $b=17$ in eq^n (1)

$a+17=5$

$\therefore$ $\boxed{a=-12}$

$\therefore$ $a=-12$, $b=17$.

6. Find Modulus and amplitude of:

(i) $-1+i\sqrt{3}$.

Let $z=-1+i\sqrt{3}$.

On comparing with $z=a+ib$. we get $a=-1$, $-ve$, $b=\sqrt{3}$, $+ve$.

$\therefore$ (a,b) lies in IInd quadrant

$\therefore$ $\boxed{\theta=\pi-\alpha}$ —— (1).

Modulus: $r=\sqrt{a^2+b^2}=\sqrt{(-1)^2+(\sqrt{3})^2}=\sqrt{4}=\boxed{2}$

Amplitude: $\alpha=\tan^{-1}\left|\frac{b}{a}\right|=\tan^{-1}\left|\frac{\sqrt{3}}{-1}\right|=\tan^{-1}(\sqrt{3})=\frac{\pi}{3}$.

But from (1),

$\theta=\pi-\alpha=\pi-\frac{\pi}{3}=\cancel{3-\pi}\ \frac{3\pi-\pi}{3}=\boxed{\frac{2\pi}{3}}$

(ii) $\frac{1}{2}-\frac{\sqrt{3}}{2}i$

Let $z=\frac{1}{2}-\frac{\sqrt{3}}{2}i$ $\therefore$ $a=\frac{1}{2}$, $+ve$ & $b=-\frac{\sqrt{3}}{2}$, $-ve$. $\therefore$ (a,b) lies in 4th quadrant.

$r=\sqrt{a^2+b^2}=\sqrt{(\frac{1}{2})+(-\frac{\sqrt{3}}{2})^2}=\sqrt{\frac{1}{4}+\frac{3}{4}}=\sqrt{\frac{4}{4}}=\boxed{1}$

$\alpha=\tan^{-1}\left|\frac{b}{a}\right|=\tan^{-1}\left|\frac{-\sqrt{3}/2}{1/2}\right|=\tan^{-1}|\sqrt{3}|=\frac{\pi}{3}$.

$\theta=2\pi-\alpha=2\pi-\frac{\pi}{3}=\frac{6\pi-\pi}{3}=\boxed{\frac{5\pi}{3}}$

⑦ Express $-\sqrt{3}+i$ in polar form:

$z = -\sqrt{3} + i$.

$\therefore a = -\sqrt{3}$, $-ve$ & $b = 1$, $+ve$

$\therefore (a, b)$ lies in 2^{nd} quadrant

$\therefore \boxed{\theta = \pi - \alpha}$ ——— ①

<u>Modulus:</u> $r = \sqrt{a^2+b^2} = \sqrt{(-\sqrt{3})^2+1^2} = \sqrt{3+1} = \sqrt{4} = \boxed{2}$

Amplitude: $\alpha = \tan^{-1}\left|\frac{b}{a}\right| = \tan^{-1}\left(\frac{1}{-\sqrt{3}}\right) = \frac{\pi}{6}$.

But $\theta = \pi - \alpha = \pi - \frac{\pi}{6} = \frac{6\pi - \pi}{6} = \frac{5\pi}{6}$

$\therefore \boxed{\theta = \frac{5\pi}{6}}$

Polar form: $z = r[\cos\theta + i\sin\theta]$

$z = 2\left[\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right]$

De Moivre's Theorem:

Solve using De Moivre's Theorem.

(1) $\dfrac{(\cos 3\theta + i\sin 3\theta)^4 (\cos 4\theta + i\sin 4\theta)^2}{(\cos 4\theta - i\sin 4\theta)^3 (\cos 5\theta - i\sin 5\theta)^{-4}}$

Let $z = \dfrac{(\cos 3\theta + i\sin 3\theta)^4 (\cos 4\theta + i\sin 4\theta)^2}{(\cos 4\theta - i\sin 4\theta)^3 (\cos 5\theta - i\sin 5\theta)^{-4}}$

$$= \frac{(\cos\theta + i\sin\theta)^{4\times 3} (\cos\theta + i\sin\theta)^{2\times 4}}{(\cos\theta + i\sin\theta)^{3\times(-4)} (\cos\theta + i\sin\theta)^{(-4)\times(-5)}}$$

$$= \frac{(\cos\theta + i\sin\theta)^{12} (\cos\theta + i\sin\theta)^{8}}{(\cos\theta + i\sin\theta)^{-12} (\cos\theta + i\sin\theta)^{20}}$$

$$= (\cos\theta + i\sin\theta)^{12+8+12-20}$$

$$= (\cos\theta + i\sin\theta)^{8}$$

$$z = \cos 8\theta + i\sin 8\theta$$

(2) $\left[\dfrac{1+\cos\alpha + i\sin\alpha}{1+\cos\alpha - i\sin\alpha}\right]^4$

Let $z = \left[\dfrac{1+\cos\alpha + i\sin\alpha}{1+\cos\alpha - i\sin\alpha}\right]^4$

$$= \frac{\left[2\cos^2\alpha/2 + i\,2\sin\alpha/2 \cdot \cos\alpha/2\right]^4}{\left[2\cos^2\alpha/2 - i\,2\sin\alpha/2 \cdot \cos\alpha/2\right]^4}$$

$$= \frac{\left[2\cos\alpha/2\,(\cos\alpha/2 + i\sin\alpha/2)\right]^4}{\left[2\cos\alpha/2\,(\cos\alpha/2 - i\sin\alpha/2)\right]^4}$$

$$= \frac{(\cos\alpha/2 + i\sin\alpha/2)^4}{(\cos\alpha/2 + i\sin\alpha/2)^{-4}} = (\cos\alpha/2 + i\sin\alpha/2)^{4+4}$$

$$= (\cos\alpha/2 + i\sin\alpha/2)^{8}$$

$$= \cos\frac{8\alpha}{2} + i\sin\frac{8\alpha}{2}$$

$$\boxed{z = \cos 4\alpha + i\sin 4\alpha}$$

③ Prove that $(\sqrt{3}+i)^{14} + (\sqrt{3}-i)^{14} = 2^{14}$.

Let $z_1 = \sqrt{3}+i$

$\therefore r_1 = \sqrt{(\sqrt{3})^2+1} = \sqrt{4}$

$\boxed{r_1 = 2}$

$\theta = \alpha = \tan^{-1}\left|\frac{1}{\sqrt{3}}\right| = \frac{\pi}{6}$

$\therefore \boxed{\theta = \frac{\pi}{6}}$

Polar form :

$z = 2\left[\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right]$

$z_1^{14} = \left[2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)\right]^{14}$

Let $z_2 = \sqrt{3} - i$

$\therefore r_2 = \sqrt{(\sqrt{3})^2+(-1)^2} = \sqrt{4} = \boxed{2}$

$\therefore \boxed{r_2 = 2}$

$\theta = -\alpha = -\tan^{-1}\left|\frac{-1}{\sqrt{3}}\right| = -\frac{\pi}{6}$.

$\therefore \boxed{\theta = -\frac{\pi}{6}}$

Polar form :

$z_2 = 2\left[\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right]$.

$z_2^{14} = \left[2\left(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right)\right]^{14}$.

$$LHS = (\sqrt{3}+i)^{14} + (\sqrt{3}-i)^{14}$$

$$= z_1^{14} + z_2^{14}$$

$$= \left[2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)\right]^{14} + \left[2\left(\cos\pi/6 + i\sin\pi/6\right)\right]^{14}$$

$$= 2^{14}\left[\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)^{14} + \left(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right)^{14}\right]$$

$$= 2^{14}\left[\cos\frac{14\pi}{6} + i\sin\frac{14\pi}{6} + \cos\frac{14\pi}{6} - i\sin\frac{14\pi}{6}\right]$$

$$= 2^{14}\left[2\cos\frac{14\pi}{6}\right]$$

$$= 2^{14}\left[2\cos\frac{7\pi}{3}\right]$$

$$= 2^{14}\left[2\cos\left(2\pi + \frac{\pi}{3}\right)\right]$$

$$= 2^{14}\left[2\cos\frac{\pi}{3}\right]$$

$$= 2^{14}\left[\cancel{2}\cdot\frac{1}{\cancel{2}}\right] \text{ — } \because \left[\cos\frac{\pi}{3} = \frac{1}{2}\right]$$

$$= 2^{14}$$

Roots of Complex Numbers

① Find the cube root of i.

Let $z = i$

$\Rightarrow z = 0 + i$, $a = 0, b = 1$.

$\therefore r = \sqrt{0^2+1^2} = \boxed{1}$

$\alpha = \tan^{-1}\left|\frac{b}{a}\right| = \tan^{-1}\left|\frac{1}{0}\right| = \tan^{-1}(\infty) = \frac{\pi}{2}$.

$\therefore \boxed{\theta = \alpha = \frac{\pi}{2}}$

Polar form: $z = r[\cos\theta + i\sin\theta]$

$z = 1[\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}]$.

General Polar form: $z[\cos(2\pi k+\theta) + i\sin(2\pi k+\theta)]$

$\therefore z = 1\left[\cos\left(2\pi k+\frac{\pi}{2}\right) + i\sin\left(2\pi k+\frac{\pi}{2}\right)\right]$

i.e. $z = \left[\cos\left(\frac{4\pi k+\pi}{2}\right) + i\sin\left(\frac{4\pi k+\pi}{2}\right)\right]$

$z^{1/3} = \left[\cos\left(\frac{4\pi k+\pi}{2}\right) + i\sin\left(\frac{4\pi k+\pi}{2}\right)\right]^{1/3}$.

By De Moivre's Th^m:

$z^{1/3} = \left[\cos\left(\frac{4\pi k+\pi}{6}\right) + i\sin\left(\frac{4\pi k+\pi}{6}\right)\right]$ —— ①

Put $k = 0, 1, 2$ in eq^n ①

$z_1 = \cos\left(\frac{4\pi(0)+\pi}{6}\right) + i\sin\left(\frac{4\pi(0)+\pi}{6}\right) = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6}$

$\boxed{z_1 = \frac{\sqrt{3}}{2} + i\frac{1}{2}}$

$z_2 = \cos\left(\frac{4\pi+\pi}{6}\right) + i\sin\left(\frac{4\pi+\pi}{6}\right) = \cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}$

$= \cos\left(\pi-\frac{\pi}{6}\right) + i\sin\left(\pi-\frac{\pi}{6}\right) = -\cos\frac{\pi}{6} + i\sin\pi/6$

$\boxed{z_2 = -\frac{\sqrt{3}}{2} + i\frac{1}{2}}$

$z_3 = \cos\left(\frac{8\pi+\pi}{6}\right) + i\sin\left(\frac{8\pi+\pi}{6}\right) = \cos\frac{9\pi}{6} + i\sin 9\pi/6$

$= \cos\left(\frac{3\pi}{2}\right) + i\sin\left(\frac{3\pi}{2}\right) = 0 - i = \boxed{-i}$

② Find all the roots of $(1+i\sqrt{3})^{1/3}$.

Let $z=(1+i\sqrt{3})$ ∴ $a=1, +ve$, $b=\sqrt{3}, +ve$ Hence (a,b) lies in I^{st} quadrant

$r=\sqrt{a^2+b^2}=\sqrt{1+3}=\sqrt{4}$ ∴ $\boxed{\theta=\alpha}$ —— ①

∴ $\boxed{r=2}$

$\alpha=\tan^{-1}\left|\frac{b}{a}\right|=\tan^{-1}\left|\frac{\sqrt{3}}{1}\right|=\tan^{-1}(\sqrt{3})=\frac{\pi}{3}$.

But from ①, $\boxed{\theta=\alpha=\frac{\pi}{3}}$

Polar form: $z=r[\cos\theta+i\sin\theta]=2\left[\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right]$

General Polar form: $z=2\left[\cos\left(2\pi k+\frac{\pi}{3}\right)+i\sin\left(2\pi k+\frac{\pi}{3}\right)\right]$

$z^{1/3}=2^{1/3}\left[\cos\left(2\pi k+\frac{\pi}{3}\right)+i\sin\left(2\pi k+\frac{\pi}{3}\right)\right]^{1/3}$.

By De. Moivre's Thm we have,

$z^{1/3}=2^{1/3}\left[\cos\frac{1}{3}\left(\frac{6\pi k+\pi}{3}\right)+i\sin\frac{1}{3}\left(\frac{6\pi k+\pi}{3}\right)\right]$

∴ $z^{1/3}=2^{1/3}\left[\cos\left(\frac{6\pi k+\pi}{9}\right)+i\sin\left(\frac{6\pi k+\pi}{9}\right)\right]$ —— ①

Put $k=0,1,2$ in eqn ③

$z_1=2^{1/3}\left[\cos\frac{\pi}{9}+i\sin\frac{\pi}{9}\right]$

$z_2=2^{1/3}\left[\cos\frac{7\pi}{9}+i\sin\frac{7\pi}{9}\right]$

$z_3=2^{1/3}\left[\cos\frac{13\pi}{9}+i\sin\frac{13\pi}{9}\right]$

Functions

Value of a Function

1. If $f(x) = x^2 - x + 1$, find $f(0)$, $f(-1)$.

$\longrightarrow$ Given $f(x) = x^2 - x + 1$ ——— (1)

Put $x = 0$ in (1).

$f(0) = 0^2 - 0 + 1 = \boxed{1}$ $\quad \therefore \boxed{f(0) = 1}$

Put $x = -1$ in eqⁿ (1)

$f(-1) = (-1)^2 - (-1) + 1 = 1 + 1 + 1 = 3.$

$\therefore \boxed{f(-1) = 3}$

(2) If $f(x) = x^3 - 3x^2 + 5$, find $f(0) + f(3)$

$\longrightarrow$ Given $f(x) = x^3 - 3x^2 + 5$ ——— (1)

$\therefore \quad f(0) = 0^3 - 3(0)^2 + 5 = \boxed{5}$

$f(3) = (3)^3 - 3(3)^2 + 5 = \boxed{5}$

$\therefore f(0) + f(3) = 5 + 5 = \boxed{10}$

(3) If $f(x) = (64)^x + \log_3 x$, find $f(\frac{1}{3})$

$\longrightarrow$ Given $f(x) = (64)^x + \log_3 x$ ——— (1)

Put $x = 1/3$ in eqⁿ (1),

$f(\frac{1}{3}) = (64)^{1/3} + \log_3 (\frac{1}{3})$

$= 4 + \log_3 (3^{-1})$

$= 4 - \log_3 3$

$= 4 - 1$

$\boxed{f(\frac{1}{3}) = 3}$

(*) Show that $f(x) = 4x^4 + 3\cos x + x\sin x + 1$ is even function

$\longrightarrow$ Given: $f(x) = 4x^4 + 3\cos x + x\sin x + 1$ —— (1)

Put $x = -x$ in eqⁿ (1).

$$f(-x) = 4(-x)^4 + 3\cos(-x) + (-x)\sin(-x) + 1$$

$$\therefore f(-x) = 4x^4 + 3\cos x + x\sin x + 1$$

$$\therefore f(-x) = f(x).$$

Hence the given function is even.

(*) Show that $f(x) = x^3 + 3\sin x + x$ is an odd function

$\longrightarrow$ Given: $f(x) = x^3 + 3\sin x + x$ —— (1)

Put $x = -x$ in eqⁿ (1)

$$f(-x) = (-x)^3 + 3\sin(-x) + (-x)$$

$$= -x^3 - 3\sin x - x$$

$$= -\left[x^3 + 3\sin x + x\right]$$

$$\therefore f(-x) = -f(x).$$

Hence the given function is odd.

(*) If $f(x) = \frac{x+2}{4x-3}$ and $t = \frac{2+3x}{4x-1}$, S.T. $f(t) = x$

Given: $f(x) = \frac{x+2}{4x-3}$ —— (1) & $t = \frac{2+3x}{4x-1}$ —— (2)

Put $x = t$ in eqⁿ (1)

$$f(t) = \frac{t+2}{4t-3} = \frac{\left[\frac{2+3x}{4x-1}\right]+2}{4\left[\frac{2+3x}{4x-1}\right]-3}$$

$$= \frac{\frac{2+3x+2(4x-1)}{\cancel{4x-1}}}{\frac{4(2+3x)-3(4x-1)}{\cancel{4x-1}}} = \frac{2+3x+8x-2}{8+12x-12x+3} = \frac{\cancel{11}x}{\cancel{11}}$$

$$\therefore \boxed{f(t) = x}$$

(*) If $f(x) = \frac{3x+4}{5x-7}$ and $g(x) = \frac{7x+4}{5x-3}$, show that $[g\circ f](x) = x$.

$\longrightarrow$ Given $f(x) = \frac{3x+4}{5x-7}$ —— ① and $g(x) = \frac{7x+4}{5x-3}$ —— ②

$\therefore [g\circ f](x) = g[f(x)]$

$= \frac{7f(x)+4}{5f(x)-3}$

$= \frac{7\left[\frac{3x+4}{5x-7}\right]+4}{5\left[\frac{3x+4}{5x-7}\right]-3}$

$= \frac{21x+\cancel{28}+20x-\cancel{28}}{\cancel{15x}+20-\cancel{15x}+21}$

$= \frac{\cancel{41}x}{\cancel{41}}$

$\therefore \boxed{[g\circ f](x) = x}$

(*) If $f(x) = x^2-4x+11$, solve the equation $f(x) = f(3x-1)$

$\longrightarrow$ Given : $f(x) = x^2-4x+11$ —— ①

$\therefore \quad f(3x-1) = (3x-1)^2-4(3x-1)+11$

$= 9x^2-6x+1-12x+4+11$

$f(3x-1) = 9x^2-18x+16$ —— ②

From ① & ②

$f(x) = f(3x-1)$.

$x^2-4x+11 = 9x^2-18x+16$

i.e. $9x^2-18x+16-x^2+4x-11 = 0$

$8x^2-14x+5 = 0$.

$\therefore 8x^2-4x-10x+5 = 0$

$4x(2x-1)-5(2x-1) = 0$

$(4x-5)(2x-1) = 0$

$\therefore 4x-5 = 0, \ 2x-1 = 0$ $\quad \therefore \boxed{x = \frac{5}{4}}$ or $\boxed{x = \frac{1}{2}}$

Numerical Solution of Algebraic Equations

I] Bisection Method :

1. Show that the root of the eqⁿ $x^3 - 9x + 1 = 0$ lies betⁿ $(2,3)$. Obtain the root by Bisection Method.

$\longrightarrow$ Let $f(x) = x^3 - 9x + 1$

$\therefore f(2) = 2^3 - 9(2) + 1 = -9 < 0$

$f(3) = 3^3 - 9(3) + 1 = 1 > 0$

$\therefore$ Root lies betⁿ $(2, 3)$

Iteration :-	a (−ve)	b (+ve)	$x = \frac{a+b}{2}$	f(x).
1	2	3	2.5	−5.875
2	2.5	3	2.75	−0.7031
3	2.75	3	2.875	−

2. Using Bisection Method, find the approximate root of $\sqrt{10}$ by performing two iteration.

$\longrightarrow$ Let $x = \sqrt{10}$

$\therefore x^2 = 10$

$\Rightarrow x^2 - 10 = 0.$

Let $f(x) = x^2 - 10.$

$\therefore f(3) = 3^2 - 10 = -1 < 0$

$f(4) = 4^2 - 10 = 6 > 0.$

$\therefore$ Root lies betⁿ $(3, 4)$

Iteration	a (−ve)	b (+ve)	$x = \frac{a+b}{2}$	f(x)
1	3	4	$x_1 = 3.5$	2.25
2	3	3.5	$x_2 = 3.25$	−

II] False Position Method :-

1. Verify that the equation $x^3-9x+1=0$ has root in (2,3). Find approximate to this root using the method of Regula False Position. [carry out two iteration]

$\rightarrow$ Let $f(x) = x^3-9x+1=0$

$\therefore f(2) = (2)^3-9(2)+1 = -9, < 0$

$f(3) = (3)^3-9(3)+1 = 1, > 0$

$\therefore$ Root lies betn (2, 3).

Iteration	a (-ve)	b (+ve)	f(a)	f(b)	$x = \frac{af(b)-bf(a)}{f(b)-f(a)}$	f(x).
1	2	3	-9	1	$x_1 = 2.9$	-0.711
2	2.9	3	-0.711	1	$x_2 = 2.941$	-

2. Using false position method, find the approx. value of $\sqrt{12}$ in the interval (3,4) by performing two iteration.

$\rightarrow$ Let $x = \sqrt{12}$

$\therefore x^2 = 12$

$\Rightarrow x^2-12 = 0$.

Let $f(x) = x^2-12$

$\therefore f(3) = (3)^2-12 = -3 < 0$

$f(4) = (4)^2-12 = 4 > 0$.

$\therefore$ Root lies betn (3,4).

Iteration	a (-ve)	b (+ve)	f(a)	f(b)	$x = \frac{af(b)-bf(a)}{f(b)-f(a)}$	f(x)
1	3	4	-3	4	$x_1 = 3.428$	-0.3036
2	3.428	4	-0.3036	4	$x_2 = 3.468$	

Newton Raphson Method :

1. Using Newton Raphson Method, find the root of equations $x^3 - 5x + 3 = 0$ in $(0,1)$ by performing two iteration.

$\longrightarrow$ Let $f(x) = x^3 - 5x + 3$.

$\therefore f'(x) = 3x^2 - 5$.

$f(0) = 0^3 - 5(0) + 3 = 3 > 0$

$f(1) = 1^3 - 5(1) + 3 = -1 < 0$.

$\therefore$ Root lies betn $(0,1)$.

Take initial root $\boxed{x_0 = 1}$

By Newton Raphson method we have,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{(x_n^3 - 5x_n + 3)}{3x_n^2 - 5}$$

$$= \frac{x_n(3x_n^2 - 5) - (x_n^3 - 5x_n + 3)}{3x_n^2 - 5}$$

$$= \frac{3x_n^3 - \cancel{5x_n} - x_n^3 + \cancel{5x_n} - 3}{3x_n^2 - 5}$$

$$\boxed{x_{n+1} = \frac{2x_n^3 - 3}{3x_n^2 - 5}} \qquad \text{——— } (1)$$

Ist iteration :— Put $n = 0$ in eqn (1).

$$x_{0+1} = \frac{2x_0^3 - 3}{3x_0^2 - 5} = \frac{2(1)^3 - 3}{3(1)^2 - 5} = \frac{-1}{-2} = \boxed{0.5}$$

$\therefore \boxed{x_1 = 0.5}$

IInd iteration :- Put $n = 1$ in eqn (1).

$$x_{1+1} = \frac{2x_1^3 - 3}{3x_1^2 - 5} = \frac{2(0.5)^3 - 3}{3(0.5)^2 - 5} = \boxed{0.647}$$

$\therefore \boxed{x_2 = 0.647}$

② Use Newton Raphson Method to evaluate the foll^n correct to three places of decimals.

(i) $\sqrt[3]{20}$ (ii) $\frac{1}{\sqrt{12}}$

Sol^n: (i) Let $x = \sqrt[3]{20}$ $\therefore x^3 = 20 \Longrightarrow x^3 - 20 = 0$.

Let $f(x) = x^3 - 20$.

$\therefore f'(x) = 3x^2$

As $\sqrt[3]{27} = 3$ and 27 is close to 20, the initial approximate is $\boxed{x_0 = 3}$

According to Newton's Raphson Method we have,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{(x_n^3 - 20)}{3x_n^2}$$

$$= \frac{3x_n^3 - x_n^3 + 20}{3x_n^2}$$

$$\therefore \boxed{x_{n+1} = \frac{2x_n^3 + 20}{3x_n^2}}$$

Put $n = 0, 1, 2, 3$ in eq^n ① we get

$$x_1 = \frac{2x_0^3 + 20}{3x_0^2} = \frac{2(3)^3 + 20}{3(3)^2} = \frac{74}{27} = \boxed{2.741}$$

$$x_2 = \frac{2x_1^3 + 20}{3x_1^2} = \frac{2(2.741)^3 + 20}{3(2.741)^2} = \frac{61.187}{22.539} = \boxed{2.715}$$

$$x_3 = \frac{2x_2^3 + 20}{3x_2^2} = \frac{2(2.715)^3 + 20}{3(2.715)^2} = \frac{60.026}{22.114} = \boxed{2.714}$$

(ii) Let $x = \frac{1}{\sqrt{12}}$ $\therefore x^2 = \frac{1}{12} \Rightarrow x^2 - \frac{1}{12} = 0$.

Let $f(x) = x^2 - \frac{1}{12}$

$\therefore f'(x) = 2x$

Now $\frac{1}{\sqrt{9}} = \frac{1}{3}$ and 9 is close to 12, take initial approx. $\boxed{x_0 = \frac{1}{3}}$

By Newton Raphson Method,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \left[\frac{x_n^2 - \frac{1}{12}}{2x_n}\right]$$

$$= \frac{2x_n^2 - x_n^2 + \frac{1}{12}}{2x_n}$$

$$\boxed{\therefore x_{n+1} = \frac{x_n^2 + \frac{1}{12}}{2x_n}} \longrightarrow (1)$$

Put n= 0, 1, 2, in eqn (1) we get,

$$x_1 = \frac{x_0^2 + \frac{1}{12}}{2x_0} = \frac{\left(\frac{1}{3}\right)^2 + \frac{1}{12}}{2\left(\frac{1}{3}\right)} = \frac{\frac{1}{9} + \frac{1}{12}}{\frac{2}{3}} = \frac{\frac{4+3}{36}}{\frac{2}{3}} = \frac{7}{36} \times \frac{3}{2}$$

$$\therefore x_1 = \frac{7}{24} = \boxed{0.2917}$$

$$x_2 = \frac{x_1^2 + \frac{1}{12}}{2x_1} = \frac{(0.2917)^2 + \frac{1}{12}}{2(0.2917)} = \frac{0.1684}{0.5834} = \boxed{0.2887}$$

$$x_3 = \frac{x_2^2 + \frac{1}{12}}{2x_2} = \frac{(0.2887)^2 + \frac{1}{12}}{2(0.2887)} = \frac{0.1666}{0.5774} = \boxed{0.2885}$$